

# Critical analysis of topological charge determination in the background of center vortices in $SU(2)$ lattice gauge theory

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We analyze topological charge contributions from classical  $SU(2)$  center vortices with shapes of planes and spheres using different topological charge definitions, namely the center vortex picture of topological charge, a discrete version of  $F\tilde{F}$  in the plaquette and hypercube definitions and the lattice index theorem. For the latter the zero-modes of the Dirac operator in the fundamental and adjoint representations using both the overlap and asqtad staggered fermion formulations are investigated. We find several problems for the individual definitions and discuss the discrepancies between the different topological charge definitions. Our results show that the interpretation of lattice quantities like  $F\tilde{F}$  in the background of center vortices is rather subtle.

## I. INTRODUCTION

Since Savvidy [1] we know that the QCD-vacuum is non-trivial and has magnetic properties. In lattice QCD it was shown [2] that vortices (quantized magnetic fluxes) condense in the vacuum and compress the electric flux between quark and antiquark to a string leading to confinement. This center vortex model [3–8] seems to be a very promising candidate to explain the phenomena that dominate the infrared regime. Numerical simulations have indicated that vortices could also account for phenomena related to chiral symmetry, such as causing topological charge fluctuations and spontaneous chiral symmetry breaking (SCSB) [9–12]. In particular, [13] states that the topological charge of a gauge field can be determined from the shape of P-vortices, *i.e.* from vortex intersections and writhing points.

Center Vortices are based on a discrete gauge symmetry of the action. A non-trivial center transformation of all link variables in one time (or space) slice [2]

$$U_0(\vec{x}, t_0) \Rightarrow z U_0(\vec{x}, t_0), \quad z \in Z_N \quad (1.1)$$

leaves the action invariant. More generally, this transformation can be expressed in terms of gauge transformations on a periodic lattice

$$U_0(\vec{x}, t_0) \Rightarrow g(x, t) U_0(\vec{x}, t_0) g^\dagger(x, t+1) \quad (1.2)$$

which are periodic in the time direction only up to a  $Z_N$  transformation:

$$g(\vec{x}, t_0 + L_t) = z g(\vec{x}, t_0), \quad z \in Z_N \quad (1.3)$$

This "singular" gauge transformation is not really a gauge transformation, since it affects of course the Polyakov loop, a gauge-invariant observable. Restricting such a transformation to a finite volume of a slice, the three dimensional Dirac volume, increases the action by a surface contribution, the vortex action. However, center vortices also increase the entropy, compensating the rise of the action and give essential contributions to

the vacuum configurations [14]. The appearance of link variables close to non-trivial center elements survives the continuum limit and leads to singular gauge fields. This implies the question, whether center vortices are lattice artifacts. One could give a positive answer, if removing such lattice artifacts would not influence QCD. But it is just the opposite, removing center vortices destroys confinement and the topological charge vanishes [9]. This means center vortices are an essential ingredient of the QCD vacuum.

## II. TOPOLOGY ON THE LATTICE

In lattice calculations there is a common method to determine the topological charge  $Q$  from the integral

$$Q = -\frac{1}{16\pi^2} \int d^4x \text{Tr}[\tilde{\mathcal{F}}_{\mu\nu} \mathcal{F}_{\mu\nu}], \quad (2.1)$$

where  $F_{\mu\nu}$  is expressed in terms of the plaquette field

$$P_{\mu\nu} = U_\mu(x) U_\nu(x + \mu) U_\mu^\dagger(x + \nu) U_\nu^\dagger(x). \quad (2.2)$$

This expression is derived in the continuum from the transition between vacua with different winding numbers [15, 16]

$$Q = \int_{S_3} J_\mu d\sigma_\mu, \quad (2.3)$$

$$J_\mu = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr}(A_\nu \partial_\alpha A_\beta + 2/3 A_\nu A_\alpha A_\beta). \quad (2.4)$$

Since  $F_{\mu\nu} \tilde{F}_{\mu\nu} = \partial_\mu J_\mu$ ,  $Q$  can be re-expressed as the above volume integral (2.1). On the lattice, continuity in space is lost and it seems that one should be able to view any lattice field configuration as being a discrete copy of a smooth continuum configuration. This would always be topologically trivial, since  $F\tilde{F}$  is a total derivative. Nonzero  $Q$  may come from field configurations containing gauge singularities [17–19].

Another possibility to analyze the topology of a gauge field is given by the Atiyah-Singer index theorem [20–23]. It states that the topological charge of a gauge field configuration is proportional to the index of the Dirac operator in this gauge field background. For the overlap Dirac operator [23–25] in the fundamental representation the index is given by  $\text{ind } D[A] = n_- - n_+ = Q$ , where  $n_-$  and  $n_+$  are the number of left- and right-handed zero-modes. The adjoint version of the index theorem reads  $\text{ind } D[A] = n_- - n_+ = 2NQ = 4Q$ , where  $N = 2$  is the number of colors and the additional factor 2 is due to the fact that the fermion is in a real representation, hence the spectrum of the adjoint Dirac operator  $iD$  is doubly degenerate. As described in [26] the improved staggered operator also produces eigenmodes which can clearly be identified as zero-modes and all results in this paper show perfect agreement between the two fermion realizations, considering that the eigenvalues of the staggered fermion operator have a twofold degeneracy due to a global charge conjugation symmetry in  $SU(2)$ . We therefore have  $\text{ind } D[A] = n_- - n_+ = 2Q$  for fundamental and  $\text{ind } D[A] = n_- - n_+ = 8Q$  for adjoint (asqtad) staggered fermions. The lattice version of the index theorem is only valid as long as the gauge field is smooth enough and satisfies a so-called “admissibility” condition. It requires that the plaquette values  $U_{\mu\nu}$  are bounded close to 1, the value for very smooth gauge fields. Sufficient, but not necessary bound for the “admissibility” of the gauge field are  $\|1 - U_{\mu\nu}\| < 1/30$  [17], or  $\|1 - U_{\mu\nu}\| < [6(2 + \sqrt{2})] = 0.04882$  [27].

In this paper we discuss a smooth lattice field configuration, which fulfills the admissibility condition and shows clearly a discrepancy between the integral of  $F\tilde{F}$  and the topological charge derived from the lattice index theorem. We work with thick, spherical vortices in  $SU(2)$  lattice gauge theory and extend our analysis of the topological charge and the lattice index theorem. The problem seems to be related to the singular nature of vortex configurations questioning the  $F\tilde{F}$  definition of topological charge. In fact it is incorporated to the  $SU(2)$  nature of our spherical vortex configuration, for which also the vortex picture of topological charge fails. Here we analyze this problem in more detail and start with the discussion of plane vortices in different  $U(1)$ -subgroups.

### III. PLANE CENTER VORTICES AND TOPOLOGICAL CHARGE CONTRIBUTIONS

We investigated planar vortices parallel to two of the coordinate axes in  $SU(2)$  lattice gauge theory with respect to the location of Dirac zero-modes in [28]. The vortices are defined by links varying in a  $U(1)$  subgroup of  $SU(2)$ ,  $U_\mu = \exp(i\phi\sigma_i)$ . The direction of the flux and the orientation of the vortices are determined by the gradient of the angle  $\phi$ , which we choose as a piecewise linear function of the coordinate perpendicular to the vortex. Upon traversing a vortex sheet, the angle  $\phi$

increases or decreases by  $\pi$  within a finite thickness of the vortex. Center projection leads to a (thin) P-vortex at half the thickness [29]. If we consider these thick, planar vortices intersecting orthogonally, each intersection carries a topological charge with modulus  $|Q| = 1/2$ , whose sign depends on the relative orientation of the vortex fluxes [30]. The plaquette definition simply discretizes the continuum (Minkowski) expression of the Pontryagin index to a lattice (Euclidean) version of the topological charge definition:

$$Q = -\frac{1}{16\pi^2} \int d^4x \text{tr}[\tilde{F}_{\mu\nu}F_{\mu\nu}] \\ = -\frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\alpha\beta} \text{tr}[F_{\alpha\beta}F_{\mu\nu}] = \frac{1}{4\pi^2} \int d^4x \vec{E} \cdot \vec{B}$$

Our  $xy$ -vortices have only non-trivial  $zt$ -plaquettes, i.e. an electric field  $E_z$ , while  $zt$ -vortices bear non-trivial  $xy$ -plaquettes corresponding to a magnetic field  $B_z$ . The topological charge is then proportional to  $E_z B_z$ . If the angle  $\phi$  for different vortex sheets rotates in the same  $U(1)$  subgroup then parallel crossings give  $Q = 1/2$  and anti-parallel crossings give  $Q = -1/2$ .

Constructing the vortex sheets out of different  $U(1)$  subgroups the individual crossings do not contribute to the topological charge if the color vectors of  $E$ - and  $B$ -plaquettes are orthogonal. Maximal center gauge still identifies two vortex pairs intersecting in four space-time points, but for parallel vortex pairs the topological charge from intersection points ( $Q=2$ ) now differs from the  $F\tilde{F}$  definition. We find configurations with zero, one and even half topological charge. If  $\phi$  rotates e.g. in the  $\sigma_1$ -subgroup for the  $xy$ -vortex and in  $\sigma_2$  for the  $zt$ -vortex then  $F\tilde{F}$  gives no contribution to the topological charge. Other configurations are shown in Fig. 1.

The action density in Fig. 1d) already shows that such orthogonal color vector intersections might be suppressed due to higher action and in fact, at the intersection points we find maximally non-trivial ( $yt$ -) plaquettes

$$P_{yt} \approx (-i\sigma_k)(-i\sigma_l)i\sigma_k i\sigma_l \\ = (\sigma_k\sigma_l)(\sigma_k\sigma_l) = i\sigma_m i\sigma_m = -\mathbb{1}.$$

These rough configurations also seem to trouble the Dirac operators, which do not find any zero modes. During cooling the  $Q = 1$ - and  $Q = 1/2$ -configurations are only meta-stable and soon turn to anti-parallel vortex pairs with vanishing topological charge, whereas the original  $Q = 0$ -configuration from two parallel vortex pairs in different  $U(1)$ -subgroups prefers the  $S = 2S_{inst}$  action minimum and ends up with parallel color vectors yielding  $Q = 2$ . For admissible gauge fields (after some cooling) we also find the correct numbers of zero modes and therefore the different definitions of topological charge agree with each other. Nevertheless, for the spherical vortex we immediately guarantee the admissibility condition but still get a discrepancy.

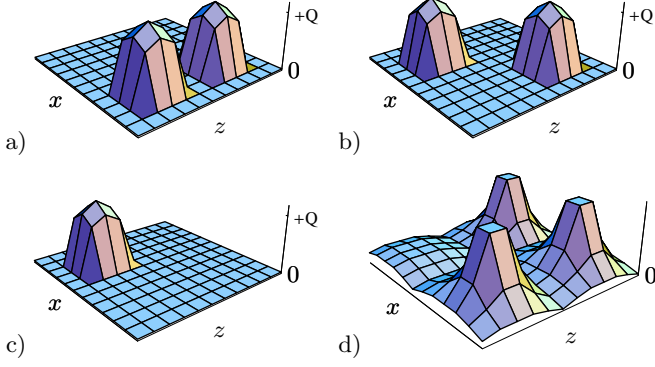


FIG. 1. Topological charge density for two parallel vortices intersecting in four points (same geometry as above). The first vortex sheet of the  $xy$ -vortex rotates  $\phi$  from zero to  $\pi$  in  $\sigma_1$  and on to  $2\pi$  in  $\sigma_2$  for the second vortex sheet, whereas the  $zt$ -vortex only rotates in  $\sigma_1$  (a) or starts with a  $\sigma_2$  rotation and rotates the second vortex sheet in  $\sigma_1$  (b) we get  $Q = 1$  - or if the second  $zt$ -vortex sheet rotates in  $\sigma_3$  we find  $Q = 1/2$  (below) distributed as shown in c) with an action density in the intersection plane shown in d).

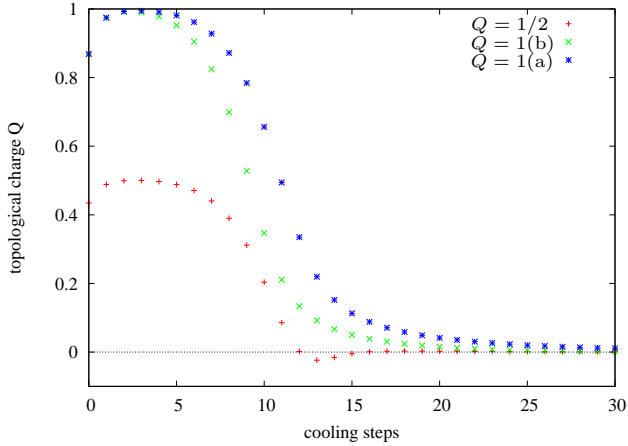


FIG. 2. Topological charge during cooling the (meta-stable)  $Q = 1$ - and  $Q = 1/2$ -configurations. The intersecting vortex sheets end up with anti-parallel orientation and the topological charge vanishes.

#### IV. THE "SPHERICAL VORTEX PROBLEM"

The spherical vortex of radius  $R$  and thickness  $\Delta$  was introduced in [31] and analyzed in more detail in [26]. It is constructed with the following links:

$$U_\mu(x_\nu) = \begin{cases} \exp(i\alpha(|\vec{r} - \vec{r}_0|)\vec{n} \cdot \vec{\sigma}) & t = 1, \mu = 4 \\ \mathbb{1} & \text{elsewhere} \end{cases} \quad (4.1)$$

$$\text{with } \vec{n}(\vec{r}, t) = \frac{\vec{r} - \vec{r}_0}{|\vec{r} - \vec{r}_0|}, \quad (4.2)$$

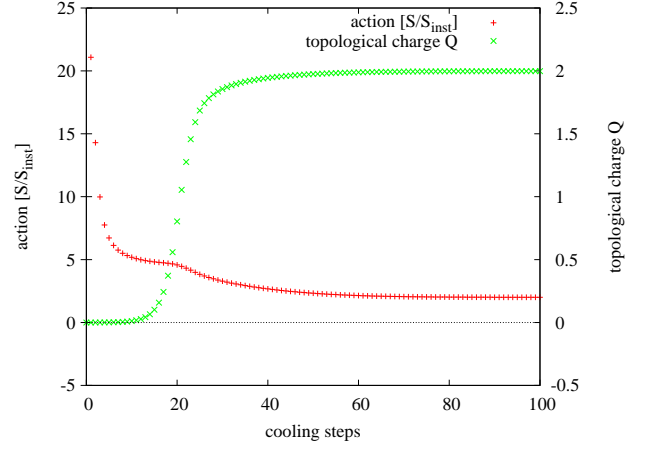


FIG. 3. During cooling the  $Q = 0$ -configuration the action reaches a plateau with  $S = 2S_{inst}$  and the parallel vortex pairs end up with parallel color vectors equivalent to the  $Q = 2$ -configuration.

where  $\vec{r}$  is the spatial part of  $x_\nu$  and the profile function  $\alpha$  is either one of  $\alpha_+, \alpha_-$ , which are defined by

$$\alpha_+(r) = \begin{cases} 0 & r < R - \frac{\Delta}{2} \\ \frac{\pi}{2} \left(1 - \frac{r-R}{\frac{\Delta}{2}}\right) & R - \frac{\Delta}{2} < r < R + \frac{\Delta}{2} \\ \pi & R + \frac{\Delta}{2} < r \end{cases}, \quad (4.3)$$

$$\alpha_-(r) = \begin{cases} \pi & r < R - \frac{\Delta}{2} \\ \frac{\pi}{2} \left(1 + \frac{r-R}{\frac{\Delta}{2}}\right) & R - \frac{\Delta}{2} < r < R + \frac{\Delta}{2} \\ 0 & R + \frac{\Delta}{2} < r \end{cases}. \quad (4.4)$$

This means that all links are equal to  $\mathbb{1}$  except for the  $t$ -links in a single time-slice at fixed  $t = 1$ . The phase changes from 0 to  $\pi$  from inside to outside (or vice versa). The graph of  $\alpha_-(r)$  is plotted in Fig. 2 in [31], giving a hedgehog-like configuration. In maximal center gauge and after center projection, this configuration shows a single, spherical vortex without any intersections and hence no topological charge. Since only links in the time direction are different from  $\mathbb{1}$ , the topological charge determined from any lattice version of  $F\tilde{F}$  vanishes for this spherical vortex configuration. The index of the considered Dirac operators however is nonzero, resulting in  $Q = \mp 1$ , for  $\alpha_\pm$ . On a  $136^3 \times N_t$  lattice the plaquettes for the spherical vortex, Eq. (4.1), satisfy the "admissibility" condition.

In order to understand the discrepancy we apply standard cooling to the spherical vortex configuration. For many cooling steps, the index of the Dirac operator does not change, but the topological charge quickly rises close to  $\mp 1$  for  $\alpha_\pm$  while the action  $S$  reaches a (nonzero) plateau. So, the index of the overlap Dirac operator agrees with the topological charge after some cooling. In Fig. 4 we plot the cooling history for a spherical vortex on a  $40^4$ -lattice. For comparison we also plot the topological charge of an instanton during cooling, which looks pretty

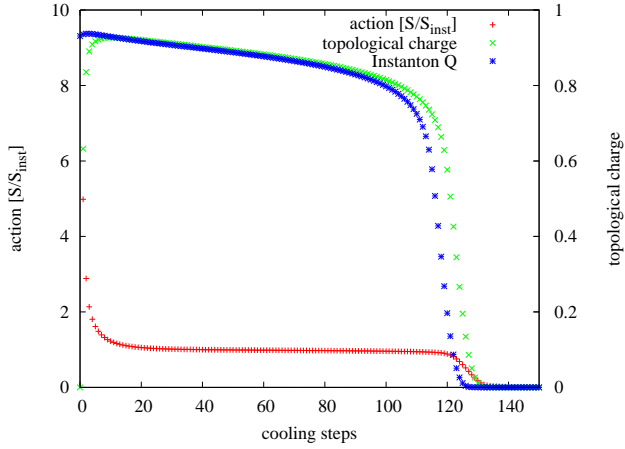


FIG. 4. Cooling of a spherical vortex on a  $40^4$  lattice. The topological charge rises from zero to close to one for  $\alpha = \alpha_-$  (right scale) while the action  $S$  (in units of the one-instanton action  $S_{\text{inst}}$ ) reaches a plateau (left scale). For comparison we also plot the topological charge during cooling of an instanton.

much the same as for the spherical vortex. In fact, the action and topological charge densities spread over more and more time-slices, developing a hyper-spherical distribution like standard instantons. We conclude that our spherical vortex develops an instanton-like structure during cooling, in agreement with [32], stating that the Hausdorff dimension of regions where the topological charge is localized, changes towards the total space dimensions.

However, the vortex structure of our initial configuration is removed after a few cooling steps, i.e. the spherical vortex shrinks very quickly. This is in agreement with the fact that a single instanton doesn't show any vortex content [33], but it clearly shows that cooling significantly changes contents of the initial gauge configuration.

Further we apply some Monte Carlo steps to our spherical vortex configuration using the Metropolis algorithm with a small spread, *i.e.* adding small quantum fluctuations, and analyze the vortex structure and topological charge. Fig. 5 shows the action and topological charge during 100 Metropolis and another 100 cooling steps. The action rises during the Monte Carlo update and the spherical vortex percolates over the whole lattice while the topological charge fluctuates around zero. The index of the lattice Dirac operator however indicates topological charge  $Q = \mp 1$  for  $\alpha_{\pm}$ , and also cooling after the Monte Carlo update still reveals the correct result for  $Q$  via  $F\tilde{F}$ , showing the same behavior as in Fig. 4.

We also want to emphasize the difference to the spherical vortex on an asymmetric lattice with time extent  $N_t = 2$ , which we analyzed in the second part of [26]. The vortex structure in one of the two time-slices survives much longer during cooling and leads to a static, singular Dirac monopole before falling through the lattice. For completeness we also plot the cooling history of a spherical vortex on a  $136^3 \times 2$  lattice in Fig. 6. The configuration initially satisfies the "admissibility" condition,

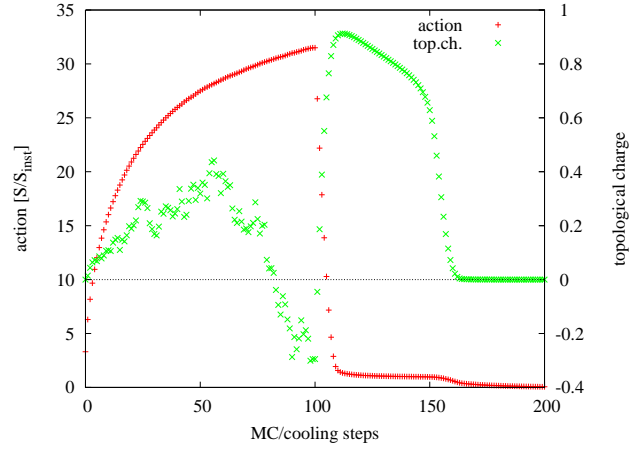


FIG. 5. Monte Carlo (Metropolis) update and cooling of a spherical vortex on a  $16^4$  lattice. Even after 100 MC steps cooling reveals the correct topological charge  $Q$  via  $F\tilde{F}$ .

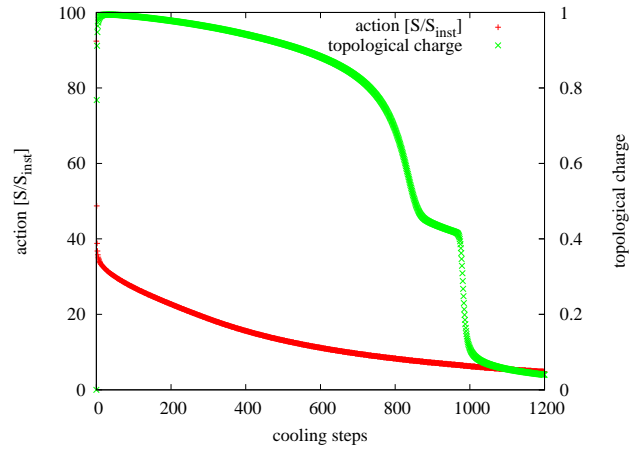


FIG. 6. Cooling of a spherical vortex on a  $136^3 \times 2$  lattice. The topological charge definition  $F\tilde{F}$  initially gives  $Q = 0$ , in discrepancy with the index of the Dirac operator. During cooling it rises to close to one for  $\alpha = \alpha_-$  resolving the discrepancy. The second plateau around 900 cooling steps indicates the fractional topological charge of a Dirac monopole, for more details see [26].

but still shows the discrepancy between  $F\tilde{F}$  and the index of the Dirac operator. The second plateau around 900 cooling steps indicates the fractional topological charge of the Dirac monopole, appearing only on the asymmetric lattice. On a symmetric lattice, we find a continuous transition to an instanton-like structure, and the vortex structure is lost. The singular gauge transformation is smoothed out in time direction at the cost of our center vortex. There seems to be no possibility of center vortices without gauge field singularities. This brings us back to the statement from section II that topological charge on the lattice may need singularities of the gauge field, which seems to agree with the fact that removal of center vortices leads to vanishing topological charge.

## V. DISCUSSION

We finally want to resolve the problem of topological charge determination via  $F\tilde{F}$  for the above discussed spherical vortex configuration. The usual expression of topological charge  $Q$  (2.1) does not take into account the periodic boundary conditions of the lattice, which modify the dynamics considerably even in the large volume limit and lead to a different topological classification [34]. The full expression for  $Q$  must contain possible twists in the boundary conditions allowed in the adjoint representation [35–37]. However, such boundary twists may also be hidden in gauge singularities, especially in singular gauge transformations defining center vortices on the lattice. Usually one tries to evaluate the integrals for gauge invariant quantities like (2.1) in the axial gauge  $A_0 = 0$ , where the singularities are transformed away [38]. However, in our particular case of the spherical vortex there is no way to gauge-transform the singularity away, except by applying the inverse of the initial singular gauge transformation, which then would define the boundary twist, giving the only contribution to the topological charge of our trivial gauge field in  $A_0 = 0$  gauge.

An easy way of thinking how to determine the topological charge of a gauge field configuration is given in [18]. Woit suggests to locate the gauge singularities, locally gauge transform them away using different gauges, and to measure the degrees of the maps relating the different gauges. The sum of these degrees will be a topological invariant, the topological charge. For our configuration we easily may follow the instructions in the above reference by splitting the lattice into time slices. We now can only consider the time slice containing our spherical vortex, where the singular gauge transformation now defines the corresponding mapping and its degree the correct topological charge.

To state the solution of our problem in a mathematical manner, we consider the homotopy of the above spherical vortex configuration, Eq. (4.1). The  $t$ -links of these spherical vortices fix the holonomy of the time-like loops, defining a map  $U_t(\vec{x}, t = 1)$  from the  $xyz$ -hyperplane at  $t = 1$  to  $SU(2)$ . Because of the periodic boundary conditions, the time-slice has the topology of a 3-torus. But, actually, we can identify all points in the exterior of the 3 dimensional sphere since the links there are trivial. Thus the topology of the time-slice is  $\mathbb{R}^3 \cup \{\infty\}$  which is homeomorphic to  $S^3$ . A map  $S^3 \rightarrow SU(2)$  is characterized by a winding number

$$N = -\frac{1}{24\pi^2} \int d^3x \epsilon_{ijk} \text{Tr}[(U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U)],$$

resulting in  $N = -1$  for positive and  $N = +1$  for negative spherical vortices. With this assignment the index of the Dirac operator and the topological charge after cooling coincide with this winding number  $N$ . Obviously such windings, given by the holonomy of the time-like loops of the spherical vortex, influence the index theorem [39, 40], which gives the correct definition of topological charge.

Other, topologically motivated lattice constructions of  $Q$  from the gauge field are given in [17] and [41], where one compares the gauge rotations necessary in contiguous cells (hypercubes) to put each cell into the same (e.g. axial) gauge. This enables one to construct transition matrices  $v_{n,\mu}$  at the lattice sites  $n$  common to neighboring cells  $c(n)$  and  $c(n + \hat{\mu})$  which can be used to derive a geometric definition of topological charge

$$Q = -\frac{1}{24\pi^2} \sum_{n \in \Lambda} \sum_{\mu, \nu, \rho, \sigma} \epsilon_{\mu\nu\rho\sigma} \{ 3 \int_{p(n, \mu, \nu)} d^2x \text{Tr}[(v_{n,\mu} \partial_\rho v_{n,\mu}^{-1})(v_{n-\hat{\mu}, \nu}^{-1} \partial_\sigma v_{n-\hat{\mu}, \nu})] + \int_{f(n, \mu)} d^3x \text{Tr}[(v_{n,\mu}^{-1} \partial_\nu v_{n,\mu})(v_{n,\mu}^{-1} \partial_\rho v_{n,\mu})(v_{n,\mu}^{-1} \partial_\sigma v_{n,\mu})],$$

where  $\Lambda$  denotes the lattice,  $p(n, \mu, \nu)$  the plaquettes and  $f(n, \mu)$  the faces (cubes) of a cell  $c(n)$ . Evaluating this expression for our spherical vortex, the only non-trivial contribution is given by the second term for  $f(n, \mu = 4)$  in the time-slice of our vortex, resulting in the expression for the winding number given above. All other terms vanish because of trivial transition functions or the vanishing of  $F\tilde{F}$  for our configuration.

## VI. CONCLUSIONS

We reported on problems defining topological charge in the background of classical center vortices on the lattice. First, planar vortex sheets are constructed by  $U(1)$  rotations in a way that they intersect orthogonally. These intersections are known to carry a topological charge  $Q = \pm 1/2$ , but if the gauge rotations are defined in different  $U(1)$ -subgroups of  $SU(2)$ ,  $F\tilde{F}$  fails to detect the topological charge contributions. The use of the Dirac operator is not save in this case because of maximally non-trivial plaquettes, which seem to suppress such configurations in the functional integral. However, for “admissible” gauge configurations of classical, spherical center vortices we find a discrepancy between  $F\tilde{F}$  and the lattice index theorem, for both, overlap and asqtad staggered fermions in the fundamental and adjoint representations. Numerically, the discrepancy equals the winding number of the spheres when they are regarded as maps  $S^3 \rightarrow SU(2)$ . The problem arises due to the periodic boundary conditions on the lattice and the fact that center vortices are based on singular gauge transformations. The standard  $F\tilde{F}$  definition of topological charge is not applicable for such gauge fields. In our case we can regard the singular gauge transformation resulting in our spherical vortex as a boundary twist, leading to an extra contribution to  $F\tilde{F}$ , resolving the discrepancy. However, this result shows that the interpretation of topological charge via  $F\tilde{F}$  is rather subtle in the background of center vortices. The mentioned admissibility conditions do not guarantee that a naive plaquette or hypercube definition

of topological charge gives the correct result. This is only guaranteed in the continuum limit, where vortex configurations may become singular and validate the derivation of topological charge via  $F\tilde{F}$ . On lattice configurations the  $F\tilde{F}$  definition of topological charge should only be used after cooling, which however may change the gauge field content significantly. Besides geometrically motivated definitions of topological charge, the most reliable tool to study the topological structure of a lattice gauge field seems to be the Dirac operator.

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